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Introduction to Fluid Dynamics*

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SUMMARY: The basic equations of fluid mechanics are stated, with enough derivation to make them plausible but without rigour. The physical meanings of the terms in the equations are explained. Again, the behaviour of fluids in real situations is made plausible, in the light of the fundamental equations, and explained in physical terms. Some applications relevant to life in the ocean are given.

Key words: Kinematics, fluid dynamics, mass conservation, Navier-Stokes equations, hydrostatics, Reynolds number, drag, lift, added mass, boundary layers, vorticity, water waves, internal waves, geostrophic flow, hydrodynamic instability.

FUNDAMENTAL LAWS AND EQUATIONS

Kinematics

What is a fluid? Specification of motion

A fluid is anything that flows, usually a liquid or a gas, the latter being distinguished by its great relative compressibility.

Fluids are treated as continuous media, and their motion and state can be specified in terms of the velocity $\mathbf{u}$, pressure p, density ρ, etc evaluated at every point in space $\mathbf{x}$ and time t. To define the density at a point, for example, suppose the point to be surrounded by a very small element (small compared with length scales of interest in experiments) which nevertheless contains a very large number of molecules. The density is then the total mass of all the molecules in the element divided by the volume of the element.

Considering the velocity, pressure, etc as functions of time and position in space is consistent with measurement techniques using fixed instruments in moving fluids. It is called the *Eulerian specification*. However, Newton's laws of motion (see below) are expressed in terms of individual particles, or fluid elements, which move about. Specifying a fluid motion in terms of the position $\mathbf{X}(t)$ of an individual particle (identified by its initial position, say) is called the *Lagrangian specification*. The two are linked by the fact that the velocity of such an element is equal to the velocity of the fluid evaluated at the position occupied by the element:

$$\frac{d\mathbf{X}}{dt} = \mathbf{u}\left[\mathbf{X}(t), t\right] . \qquad (1)$$

The path followed by a fluid element is called a *particle path*, while a curve which, at any instant, is everywhere parallel to the local fluid velocity vector

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is called a streamline. Particle paths are coincident with streamlines in steady flows, for which the velocity $\mathbf{u}$ at any fixed point $\mathbf{x}$ does not vary with time t.

Material derivative; acceleration.

Newton's Laws refer to the acceleration of a particle. A fluid element may have acceleration both because the velocity at its location in space is changing (*local acceleration*) and because it is moving to a location where the velocity is different (*convective acceleration*). The latter exists even in a steady flow.

How to evaluate the rate of change of a quantity at a moving fluid element, in the Eulerian specification? Consider a scalar such as density ρ ($\mathbf{x}$,t). Let the particle be at position $\mathbf{x}$ at time t, and move to $\mathbf{x} + \delta\mathbf{x}$ at time $t + \delta t$, where (in the limit of small δt)

$$\delta\mathbf{x} = \mathbf{u}(\mathbf{x},t)\delta t \ . \tag{2}$$

Then the rate of change of ρ following the fluid, or *material derivative*, is

$$\frac{D\rho}{Dt} = \lim_{\delta t\to 0} \frac{\rho(\mathbf{x}+\delta\mathbf{x}, t+\delta t) - \rho(\mathbf{x},t)}{\delta t}$$

$$= \frac{\partial\rho}{\partial x}\frac{\delta x}{\delta t} + \frac{\partial\rho}{\partial y}\frac{\delta y}{\delta t} + \frac{\partial\rho}{\partial z}\frac{\delta z}{\delta t} + \frac{\partial\rho}{\partial t}$$

(by the chain rule for partial differentiation)

$$= \frac{\partial\rho}{\partial t} + u\frac{\partial\rho}{\partial x} + v\frac{\partial\rho}{\partial y} + w\frac{\partial\rho}{\partial z} \tag{3a}$$

(using (2))

$$= \frac{\partial\rho}{\partial t} + \mathbf{u}.\nabla\rho \tag{3b}$$

in vector notation, where the vector $\nabla\rho$ is the gradient of the scalar field ρ :

$$\nabla\rho = \left(\frac{\partial\rho}{\partial x}, \frac{\partial\rho}{\partial y}, \frac{\partial\rho}{\partial z}\right) .$$

A similar exercise can be performed for each component of velocity, and we can write the x-component of acceleration as

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}, \tag{4a}$$

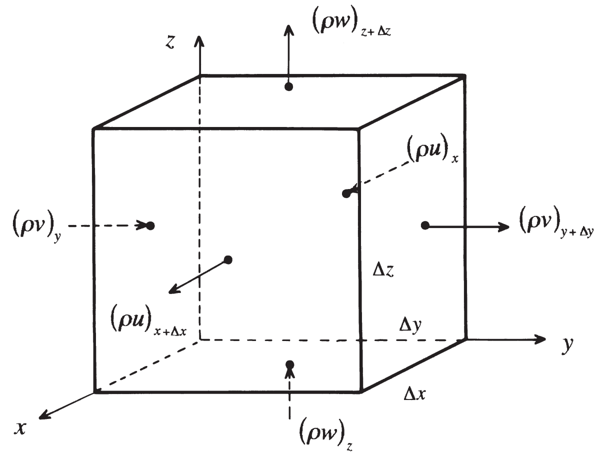

FIG. 1. – Mass flow into and out of a small rectangular region of space.

etc. Combining all three components in vector shorthand we write

$$\frac{D\mathbf{u}}{Dt} = \frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}.\nabla)\mathbf{u}, \tag{4b}$$

but care is needed because the quantity $\nabla\mathbf{u}$ is not defined in standard vector notation. Note that $\partial\mathbf{u}/\partial t$ is the local acceleration, $(\mathbf{u}.\nabla)\mathbf{u}$ the convective acceleration. Note too that the convective acceleration is *nonlinear* in $\mathbf{u}$, which is the source of the great complexity of the mathematics and physics of fluid motion.

Conservation of mass

This is a fundamental principle, stating that for any closed volume fixed in space, the rate of increase of mass within the volume is equal to the net rate at which fluid enters across the surface of the volume. When applied to the arbitrary small rectangular volume depicted in fig. 1, this principle gives:

$$\Delta x\Delta y\Delta z\frac{\partial\rho}{\partial t} = \Delta y\Delta z\left(\left[\rho u\right]_x - \left[\rho u\right]_{x+\Delta x}\right) +$$

$$+\Delta z\Delta x\left(\left[\rho v\right]_y - \left[\rho v\right]_{y+\Delta y}\right) +$$

$$+\Delta x\Delta y\left(\left[\rho w\right]_z - \left[\rho w\right]_{z+\Delta z}\right).$$

Dividing by Δx Δy Δ z and taking the limit as the volume becomes very small we get

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}(\rho u) - \frac{\partial}{\partial y}(\rho v) - \frac{\partial}{\partial z}(\rho w) \qquad (5a)$$

or (in shorthand)

$$\frac{\partial \rho}{\partial t} = -\mathrm{div}(\rho \mathbf{u}) \qquad (5b)$$

where we have introduced the *divergence* of a vector. Differentiating the products in (5a) and using (3), we obtain

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho \mathrm{div}\mathbf{u}. \qquad (6)$$

This says that the rate of change of density of a fluid element is positive if the divergence of the velocity field is negative, i.e. if there is a tendency for the flow to converge on that element.

If a fluid is *incompressible* (as liquids often are, effectively) then even if its density is not uniform everywhere (e.g. in a stratified ocean) the density of each fluid element cannot change, so

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0 \qquad (7)$$

everywhere, and the velocity field *must* satisfy

$$\mathrm{div}\mathbf{u} = \mathbf{0} \qquad (8a)$$

or

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0\ . \qquad (8b)$$

This is an important constraint on the flow of an incompressible fluid.

The Navier-Stokes equations

Newton's Laws of Motion

Newton's first two laws state that if a particle (or fluid element) has an acceleration then it must be experiencing a force (vector) equal to the product of the acceleration and the mass of the particle:

force = *mass* × *acceleration*.

For any collection of particles this becomes

net force = rate of change of momentum

where the momentum of a particle is the product of its mass and its velocity. Newton's third law states that, if two elements A and B exert forces on each other, the force exerted by A on B is the negative of the force exerted by B on A.

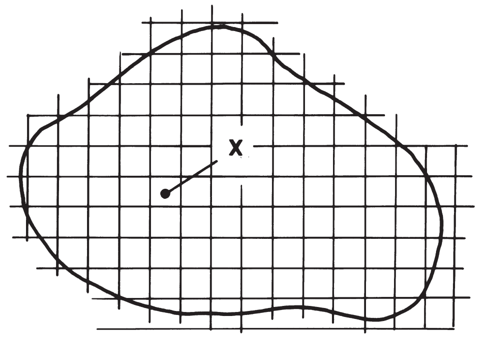

FIG. 2. – An arbitrary region of fluid divided up into small rectangular elements (depicted only in two dimensions).

To apply these laws to a region of continuous fluid, the region must be thought of as split up into a large number of small fluid elements (fig. 2), one of which, at point **x** and time t, has volume ΔV , say. Then the mass of the element is ρ (**x**,t) ΔV , and its acceleration is D**u**/Dt evaluated at (**x**,t). What is the force?

Body force and stress

The force on an element consists in general of two parts, a *body force* such as gravity exerted on the element independently of its neighbours, and *surface forces* exerted on the element by all the other elements (or boundaries) with which it is in contact. The gravitational body force on the element ΔV is $\mathbf{g}\rho$ (**x**, t) ΔV , where **g** is the gravitational acceleration. The surface force acting on a small planar surface, part of the surface of the element of interest, can be shown to be proportional to the area of the surface, ΔA say, and simply related to its orientation, as represented by the perpendicular (normal) unit vector **n** (fig. 3). The force per unit area, or *stress*, is then given by

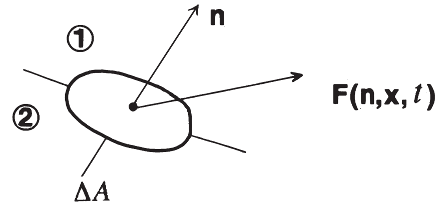

FIG. 3. – Surface force on an arbitrary small surface element embedded in the fluid, with area ΔA and normal **n**. **F** is the force exerted *by* the fluid on side 1, *on* the fluid on side 2.

$$F_x = \sigma_{xx} n_x + \sigma_{xy} n_y + \sigma_{xz} n_z$$

$$F_y = \sigma_{yx} n_x + \sigma_{yy} n_y + \sigma_{yz} n_z \qquad (9a)$$

$$F_z = \sigma_{zx} n_x + \sigma_{zy} n_y + \sigma_{zz} n_z$$

or, in shorthand,

$$\mathrm{F} = \underset{\approx}{\sigma}\, \mathbf{n} \qquad (9b)$$

where $\underset{\approx}{\sigma}$ is a matrix quantity, or *tensor*, depending on $\mathbf{x}$ and t but not $\mathbf{n}$ or ΔA. $\underset{\approx}{\sigma}$ is called the *stress tensor*, and can be shown to be *symmetric* (i.e. $\sigma_{yx} = \sigma_{xy}$, etc) so it has just 6 independent components.

It is an experimental observation that the *stress in a fluid at rest* has a magnitude independent of $\mathbf{n}$ and is always parallel to $\mathbf{n}$ and negative, i.e. compressive. This means that $\sigma_{xy} = \sigma_{yz} = \sigma_{zx} = 0$, $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -p$, say, where p is the positive *pressure* (hydrostatic pressure); alternatively,

$$\underset{\approx}{\sigma} = -p\, \underset{\approx}{\mathbf{I}} \qquad (10)$$

where $\underset{\approx}{\mathbf{I}}$ is the identity matrix.

The relation between stress and deformation rate

In a moving fluid, the motion of a general fluid element can be thought of as being broken up into three parts: translation as a rigid body, rotation as a rigid body, and deformation (see fig. 4). Quantitatively, the translation is represented by the velocity field $\mathbf{u}$, the rigid rotation is represented by the curl of the velocity field, or *vorticity*,

$$\omega = \mathrm{curl}\mathbf{u}\ , \qquad (11)$$

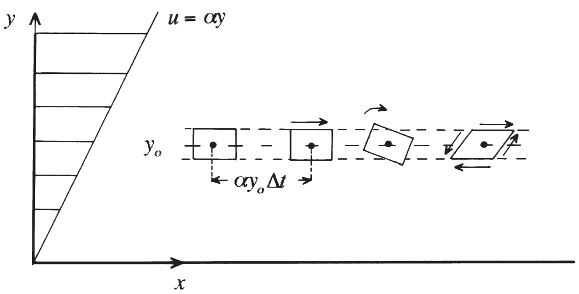

FIG. 4. – A unidirectional shear flow in which the velocity is in the x- direction and varies linearly with the perpendicular component $y : u = \alpha y$. In time Δt a small rectangular fluid element at level y_0 is *translated* a distance $\alpha y_0 \Delta t$, *rotated* through an angle $\alpha/2$, and *deformed* so that the horizontal surfaces remain horizontal, and the vertical surfaces are rotated through an angle α.

and the deformation is represented by the *rate of deformation* (or rate of strain) $\underset{\approx}{\mathbf{e}}$ which, like stress, is a symmetric tensor quantity made up of the symmetric part of the velocity gradient tensor. Formally,

$$\underset{\approx}{\mathbf{e}} = \frac{1}{2}\left(\nabla \mathbf{u} + \nabla \mathbf{u}^T\right) \qquad (12)$$

or, in full component form,

$$\underset{\approx}{\mathbf{e}} = \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{1}{2}\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) & \dfrac{1}{2}\left(\dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y}\right) & \dfrac{\partial v}{\partial y} & \dfrac{1}{2}\left(\dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial w}{\partial x} + \dfrac{\partial u}{\partial z}\right) & \dfrac{1}{2}\left(\dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z}\right) & \dfrac{\partial w}{\partial z} \end{pmatrix} \qquad (13)$$

Note that the sum of the diagonal elements of $\underset{\approx}{\mathbf{e}}$ is equal to div $\mathbf{u}$.

It is a further matter of experimental observation that, whenever there is motion in which deformation is taking place, a stress is set up in the fluid which tends to resist that deformation, analogous to friction. The property of the fluid that causes this stress is its viscosity. Leaving aside pathological ('non-Newtonian') fluids the resisting stress is generally proportional to the deformation rate. Combining this stress with pressure, we obtain the *constitutive equation for a Newtonian fluid*:

$$\underset{\approx}{\sigma} = -p\, \underset{\approx}{\mathbf{I}} + 2\mu\, \underset{\approx}{\mathbf{e}} - 2/3\mu \,\mathrm{div}\, \mathbf{u}\underset{\approx}{\mathbf{I}} \qquad (14)$$

The last term is zero in an incompressible fluid, and we shall ignore it henceforth. The quantity μ is the dynamic viscosity of the fluid.

To illustrate the concept of viscosity, consider the unidirectional shear flow depicted in fig. 4 where the plane y=0 is taken to be a rigid boundary. The normal vector $\mathbf{n}$ is in the y-direction, so equations (9) show that the stress on the boundary is

$$\mathbf{F} = \left(\sigma_{xy}, \sigma_{yy}, \sigma_{zy}\right).$$

From (14) this becomes

$$\mathbf{F} = \left(2\mu e_{xy}, -p + \mu e_{yy}, \mu e_{zy}\right),$$

but because the velocity is in the x-direction only and varies with y only, the only non-zero component of $\underset{\approx}{\mathbf{e}}$ is $e_{xy} = \frac{1}{2}\frac{\partial u}{\partial y}$. Hence

$$\mathbf{F} = \left(\mu\frac{\partial u}{\partial y}, -p, 0\right)$$

In other words, the boundary experiences a perpendicular stress, downwards, of magnitude p, the pressure, and a tangential stress, in the x-direction, equal to μ times the velocity gradient $\partial u/\partial y$. (It can be seen from (9) and (14) that tangential stresses are always of viscous origin.)

The Navier-Stokes equations

The easiest way to apply Newton's Laws to a moving fluid is to consider the rectangular block element in fig. 5. Newton's Law says that the mass of the element multiplied by its acceleration is equal to the total force acting on it, i.e. the sum of the body force and the surface forces over all six faces. The resulting equation is a vector equation; we will consider just the x-component in detail. The x-component of the stress forces on the faces perpendicular to the x-axis is the difference between the perpendicular stress σ_{xx} evaluated at the right-hand face ($x+\Delta x$) and that evaluated at the left-hand face (x) multiplied by the area of those faces, $\Delta y\Delta z$, i.e.

$$\left(\sigma_{xx}\big|_{x+\Delta x} - \sigma_{xx}\big|_{x}\right)\Delta y\Delta z.$$

If Δx is small enough, this is

$$\frac{\partial\sigma_{xx}}{\partial x}\Delta x\Delta y\Delta z.$$

The x-component of the forces on the faces perpendicular to the y-axis is

$$\left(\sigma_{xy}\big|_{y+\Delta y} - \sigma_{xy}\big|_{y}\right)\Delta z\Delta x = \frac{\partial\sigma_{xy}}{\partial y}\Delta x\Delta y\Delta z,$$

and similarly for the faces perpendicular to the z-axis. Hence the x-component of Newton's Law gives

$$\left(\rho\Delta x\Delta y\Delta z\right)\frac{\mathrm{D}u}{\mathrm{D}t} = \left(\rho g_x\right)\Delta x\Delta y\Delta z + \left(\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} + \frac{\partial\sigma_{xz}}{\partial z}\right)\Delta x\Delta y\Delta z$$

or, dividing by the element volume,

$$\rho\frac{\mathrm{D}u}{\mathrm{D}t} = \rho g_x + \frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} + \frac{\partial\sigma_{xz}}{\partial z}. \qquad (15a)$$

Similar equations arise for the y- and z-components, and they can be combined in vector form to give

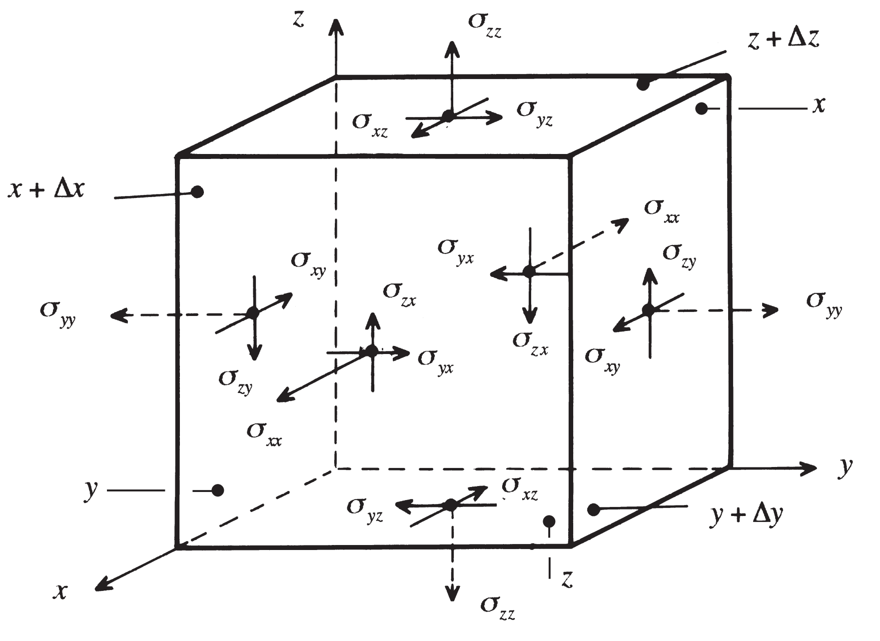

FIG. 5. – Normal and tangential surface forces per unit area (stress) on a small rectangular fluid element in motion.

$$\rho \frac{D\mathbf{u}}{Dt} = \rho\mathbf{g} + div\,\underset{\approx}{\sigma} \quad (15b)$$

The equations can be further transformed, using the constitutive equation (14) (with div $\mathbf{u} = 0$) and (13) to express $\underset{\approx}{\mathbf{e}}$ in terms of $\mathbf{u}$, to give for (15a)

$$\rho \frac{Du}{Dt} = \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right). \quad (16a)$$

Similarly in the y- and z-directions:

$$\rho \frac{Dv}{Dt} = \rho g_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \quad (16b)$$

$$\rho \frac{Dw}{Dt} = \rho g_z - \frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right). \quad (16c)$$

In these equations, it should not be forgotten that Du/Dt etc are given by equations (4).

Finally, the above three equations can be compressed into a single vector equation as follows:

$$\rho \frac{D\mathbf{u}}{Dt} = \rho\mathbf{g} - \nabla p + \mu\nabla^2\mathbf{u} \quad (16d)$$

where the symbol ∇^2 is shorthand for

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial}{\partial z^2}.$$

Equations (16a-c), or (16d), are the Navier-Stokes equations for the motion of a Newtonian viscous fluid. Recall that the left side of (16d) represents the mass-acceleration, or *inertia* terms in the equation, while the three terms on the right side are respectively the *body force*, the *pressure gradient,* and the *viscous term.*

The four equations (16a-c) and (8b) are four non-linear partial differential equations governing four unknowns, the three velocity components u,v,w, and the pressure p, each of which is in general a function of four variables, x, y, z and t. Note that if the density ρ is variable, that is a fifth unknown, and the corresponding fifth equation is (7). Not surprisingly, such equations cannot be solved in general, but they can be used as a framework to understand the physics of fluid motion in a variety of circumstances.

A particular simplification that can sometimes be made is to neglect viscosity altogether (to assume that the fluid is *inviscid*). Conditions in which this is permitted are discussed below. When it is allowed, however, we can put $\mu = 0$ in equations (16) and these are greatly simplified.

For quantitative purposes we should note the values of density and viscosity for fresh water and air at 1 atmosphere pressure and at different temperatures:

Temp	Water		Air (dry)	
	$\rho(kgm^{-3})$	$\mu(kgm^{-1}s^{-1})$	$\rho(kgm^{-3})$	$\mu(kgm^{-1}s^{-1})$
0°C	1.0000 x 10^3	1.787 x 10^{-3}	1.293	1.71 x 10^{-5}
10°C	0.9997 x 10^3	1.304 x 10^{-3}	1.247	1.76 x 10^{-5}
20°C	0.9982 x 10^3	1.002 x 10^{-3}	1.205	1.81 x 10^{-5}

Boundary conditions

Whether the fluid is viscous or not, it cannot cross the interface between itself and another medium (fluid or solid), so the normal component of velocity of the fluid at the interface must equal the normal component of the velocity of the interface itself:

$$u_n = U_n \text{ or } \mathbf{n}.\mathbf{u} = \mathbf{n}.\mathbf{U} \quad (17a)$$

where $\mathbf{U}$ is the interface velocity. In particular, on a solid boundary at rest,

$$\mathbf{n}.\mathbf{u} = 0 \quad (17b)$$

In a viscous fluid it is another empirical fact that the velocity is continuous everywhere, and in particular that the *tangential* component of the velocity of the fluid at the interface is equal to that of the interface - the *no-slip condition.* Hence

$$\mathbf{u} = \mathbf{U} \quad (18)$$

at the interface ($\mathbf{u} = 0$ on a solid boundary at rest).

There are boundary conditions on stress as well as on velocity. In general they can be summarised by the statement that the stress $\mathbf{F}$ (eq.9) must be continuous across every surface (not the stress tensor, note, just $\underset{\approx}{\sigma}.\mathbf{n}$), a condition that follows from Newton's third law. At a solid boundary this condition tells you what the force per unit area is and the total stress force on the boundary as a whole is obtained by integrating the stress over the boundary (thus the total force exerted by the fluid on an immersed solid body can be calculated).

When the fluid of interest is water, and the boundary is its interface with the air, the dynamics of the air can often be neglected and the atmosphere can be thought of as just exerting a pressure on the liquid. Then the boundary conditions on the liquid's motion are that its pressure (modified by a small viscous normal stress) is equal to atmospheric pressure and that the viscous shear stress is zero.

CONSEQUENCES: PHYSICAL PHENOMENA

Hydrostatics

We consider a fluid at rest in the gravitational field, with a free upper surface at which the pressure is atmospheric. We choose a coordinate system *x, y, z* such that z is measured vertically upwards, so $g_x = g_y = 0$ and $g_z = -g$, and we choose $z = 0$ as the level of the free surface. The density ρ may vary with height, z. Thus all components of **u** are zero, and pressure $p = p_{atm}$ at $z = 0$. The Navier-Stokes equations (16) reduce simply to

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = 0, \ \frac{\partial p}{\partial z} = -\rho g.$$

Hence

$$p = p_{atm} + g\int_z^0 \rho dz \tag{19}$$

or, for a fluid of constant density,

$$p = p_{atm} - g\rho z :$$

the pressure increases with depth below the free surface (z increasingly negative).

The above results are independent of whether there is a body at rest submerged in the fluid. If there is, one can calculate the total force exerted by the fluid by integrating the pressure, multiplied by the appropriate component of the normal vector **n**, over the body surface. The result is that, whatever the shape of the body, the net force is an upthrust and equal to g times the mass of fluid displaced by the body. This is *Archimedes' principle*. If the fluid density is uniform, and the body has uniform density ρ_b, then the net force on the body, gravitational and upthrust, corresponds to a downwards force equal to

$$(\rho_b - \rho)Vg \tag{20}$$

where V is the volume of the body. The quantity $(\rho_b - \rho)$ is called the *reduced density* of the body.

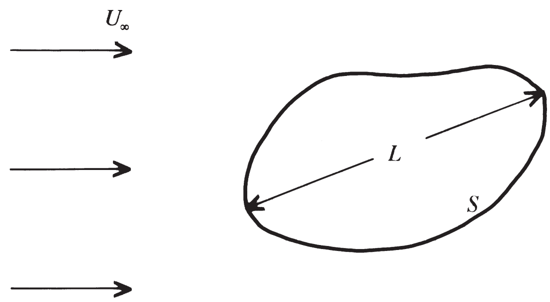

FIG. 6. – Flow of a uniform stream with velocity U_∞ in the x-direction past a body with boundary S which has a typical length scale L.

Note that, for constant density problems in which the pressure does not arise explicitly in the boundary conditions (e.g. at a free surface), the gravity term can be removed from the equations by including it in an *effective pressure*, p_e. Put

$$p_e = p + g\rho z \tag{21}$$

in equations (16) (with $g_x = g_y = 0$, $g_z = -g$) and see that g disappears from the equations, as long as p_e replaces p.

Flow past bodies

The flow of a homogeneous incompressible fluid of density ρ and viscosity μ past bodies has always been of interest to fluid dynamicists in general and to oceanographers or ocean engineers in particular. We are concerned both with fixed bodies, past which the flow is driven at a given speed (or, equivalently, bodies impelled by an external force through a fluid otherwise at rest) and with self-propelled bodies such as marine organisms.

Non-dimensionalisation: the Reynolds number

Consider a fixed rigid body, with a typical length scale L, in a fluid which far away has constant, uniform velocity U_∞ in the x-direction (fig. 6). Whenever we want to consider a particular body, we choose a sphere of radius a, diameter $L = 2a$. The governing equations are (8) and (16), and the boundary conditions on the velocity field are

$$u = v = w = 0 \quad \text{on the body surface, S} \tag{22}$$

$$u \to U_\infty, v \to 0, w \to 0 \quad \text{at infinity.} \tag{23}$$

Usually the flow will be taken to be steady, ie $\frac{\partial}{\partial t} \equiv 0$, but we shall also wish to think about development of the flow from rest.

For a body of given shape, the details of the flow (i.e. the velocity and pressure at all points in the fluid, the force on the body, etc) will depend on U_∞, L , μ and ρ as well as on the shape of the body. However, we can show that the flow in fact depends only on one dimensionless parameter, the *Reynolds number*

$$Re = \frac{\rho L U_\infty}{\mu}, \tag{24}$$

and not on all four quantities separately, so only one range of experiments (or computations) would be required to investigate the flow, not four. The proof arises when we express the equations in *dimensionless* form by making the following transformations:

$$x' = x/L,\ y' = y/L,\ z' = z/L,\ t' = U_\infty t/L,$$

$$u' = u/U_\infty,\ v' = v/U_\infty,\ w' = w/U_\infty, p' = p/\rho U_\infty^2.$$

Then the equations become: (8b):

$$\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} + \frac{\partial w'}{\partial z'} = 0; \tag{25}$$

(16a), with $\frac{\mathrm{D}u}{\mathrm{D}t}$ replaced by (4a):

$$\begin{aligned}\frac{\partial u'}{\partial t'} + u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} + w'\frac{\partial u'}{\partial z'} &= \\ = -\frac{\partial p'}{\partial x'} + \frac{1}{\mathrm{Re}}\left[\frac{\partial^2 u'}{\partial x'^2} + \frac{\partial^2 u'}{\partial y'^2} + \frac{\partial^2 u'}{\partial z'^2}\right]\end{aligned} \tag{26}$$

and there are similar equations starting with $\partial v'/\partial t'$, $\partial w'/\partial t'$. The boundary condition (22) is unchanged, though the boundary S is now non-dimensional, so its shape is important but L no longer appears. Boundary condition (23) becomes

$$u' \to 1, v' \to 0, w' \to 0 \quad \text{at infinity.} \tag{27}$$

Thus Re is the only parameter involving the physical inputs to the problem that still arises.

The drag force on the body (parallel to U_∞) proves to be of the form:

$$D = \tfrac{1}{2}\rho U^2 A C_D \tag{28}$$

where A (proportional to L^2)is the frontal area of the body ($\pi L^2/4$ for a sphere) and C_D is called the *drag coefficient*. It is a dimensionless number, computed by integrating the dimensionless stress over the surface of the body.

From now on time and space do not permit derivation of the results from the equations. Results will be quoted, and discussed physically where appropriate.

It can be seen from (26) that, in order of magnitude terms, Re represents the ratio of the nonlinear inertia terms on the left hand side of the equation to the viscous terms on the right. The flow past a rigid body has a totally different character according as Re is much less than or much greater than 1.

Low Reynolds number flow

When Re <<1, viscous forces dominate the flow and inertia is negligible. Reverting to dimensional form, the Navier-Stokes equations (16d) reduce to the Stokes equations

$$\nabla p_e = \mu \nabla^2 \mathbf{u}\ , \tag{29}$$

where gravity has been incorporated into p_e using eq. (21). The conservation of mass equation div $\mathbf{u}$ = 0, is of course unchanged. Several important conclusions can be deduced from this linear set of equations (and boundary conditions).

(i) **Drag** The force on the body is linearly related to the velocity and the viscosity: thus, for example, the drag is given by

$$D = k\mu U_\infty L \tag{30}$$

for some dimensionless constant k (thus the drag coefficient C_D is inversely proportional to Re). In particular, for a sphere of radius a, $k = 3\pi$, so

$$D = 6\pi\mu U_\infty a \tag{31}$$

It is interesting to note that the pressure and the viscous shear stress on the body surface contribute comparable amounts to the drag. The net gravitational force on a sedimenting sphere of density ρ_b, from (20), is $(\rho_b-\rho)\cdot 4/3\pi a^3 g$. This must be balanced by the drag, $6\pi U_s a$, where U_s is the sedimentation speed. Equating the two gives

$$U_s = \frac{2}{9}\frac{(\rho_b - \rho) g a^2}{\mu} \tag{32}$$

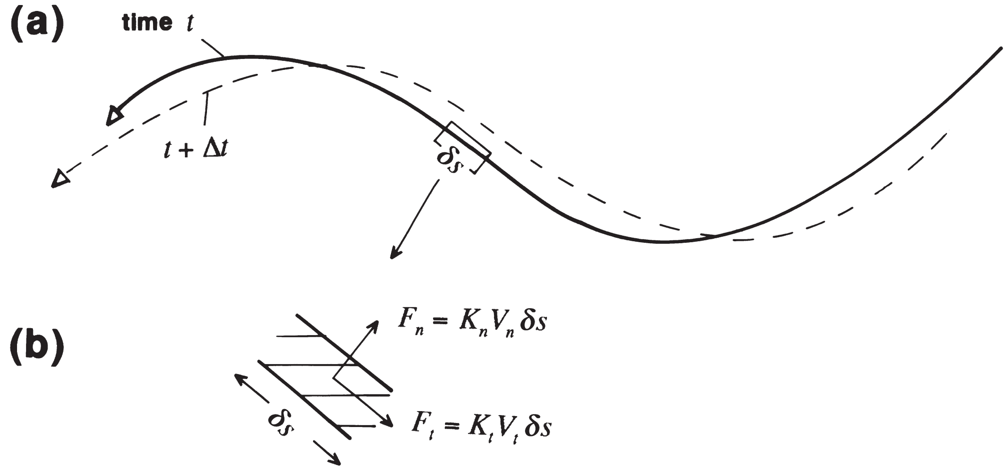

FIG. 7. – (**a**) Sketch of a swimming spermatozoon, showing its position at two successive times and indicating that, while the organism swims to its left, the wave of bending on its flagellum propagates to the right. (**b**) Blow up of a small element δs of the flagellum indicating the force components normal and tangential to it, proportional to the normal and tangential components of relative velocity.

For example, a sphere of radius 10 μm, with density 10% greater than water (ρ=10^3kg m^{-3}, $\mu\approx$11kg m^{-1}s^{-1}) will sediment out at only 20 μm^{-1}, whereas if the radius is 100 μm, the sedimentation speed will be 2 mms^{-1}.

(ii) **Quasi-steadiness**. Because the $\partial/\partial t$ term in the equations vanishes at low Reynolds number, it is immaterial whether the relative velocity of the body (or parts of it) and the fluid is steady or not. The flow at any instant is the same as if the boundary motions at that instant had been maintained steadily for a long time - i.e. the flow (and the drag force etc) is *quasi-steady*.

(iii) **The far field**. It can be shown that the far field flow, that is the departure of the velocity field from the uniform stream U_∞, dies off very slowly as the distance r from an origin inside the body becomes large. In fact it dies off as $1/r$, much more slowly, for example, than the inverse square law of Newtonian gravitation or electrostatics. This has an important effect on particle - particle interactions in suspensions. Moreover, this far field flow is proportional to the net force vector $-\mathbf{D}$ exerted *by* the body *on* the fluid, independent of the shape of the body. Thus, in vector form, we can write

$$\mathbf{u} - \mathbf{U}_\infty \approx \frac{1}{r}\left[\mathbf{P} + \frac{(\mathbf{P}.\mathrm{x})\mathrm{x}}{r^2}\right] \tag{33}$$

where

$$\mathbf{P} = \frac{-\mathbf{D}}{8\pi\mu}. \tag{34}$$

Measuring the far field is therefore one potential way of estimating the force on the body.

The only exception to the above is the case where the net force on the body (or fluid) is zero, as for a *neutrally buoyant*, self-propelled micro-organism. In that case **P** is zero, the far field dies off like $1/r^2$, and it does depend on the shape of the body and the details of how it is propelling itself.

(iv) **Uniqueness and Reversibility**. If **u** is a solution for the velocity field with a given velocity distribution $\mathbf{u}_s$ on the boundary S, then it is the only possible solution (that seems obvious, but is not true for large Re). It also follows that $-\mathbf{u}$ is the (unique) velocity field if the boundary velocities are reversed, to $-\mathbf{u}_s$. Thus if a boundary moves backwards and forwards reversibly, all elements of the fluid will also move backwards and forwards reversibly, and will not have moved, relative to the body, after a whole number of cycles. Hence a micro-organism *must* have an irreversible beat in order to swim.

(v) **Flagellar propulsion**. Many micro-organisms swim by beating or sending a wave down one or more *flagella*. Fig. 7 sketches a monoflagellate (e.g. a spermatozoon). It sends a, usually helical, wave along the flagellum from the head. This is a non-reversing motion because the wave constantly propagates along. The reason that such a wave can produce a net *thrust*, to overcome the *drag* on the head (and on the tail too) is that about twice as much force is generated by a

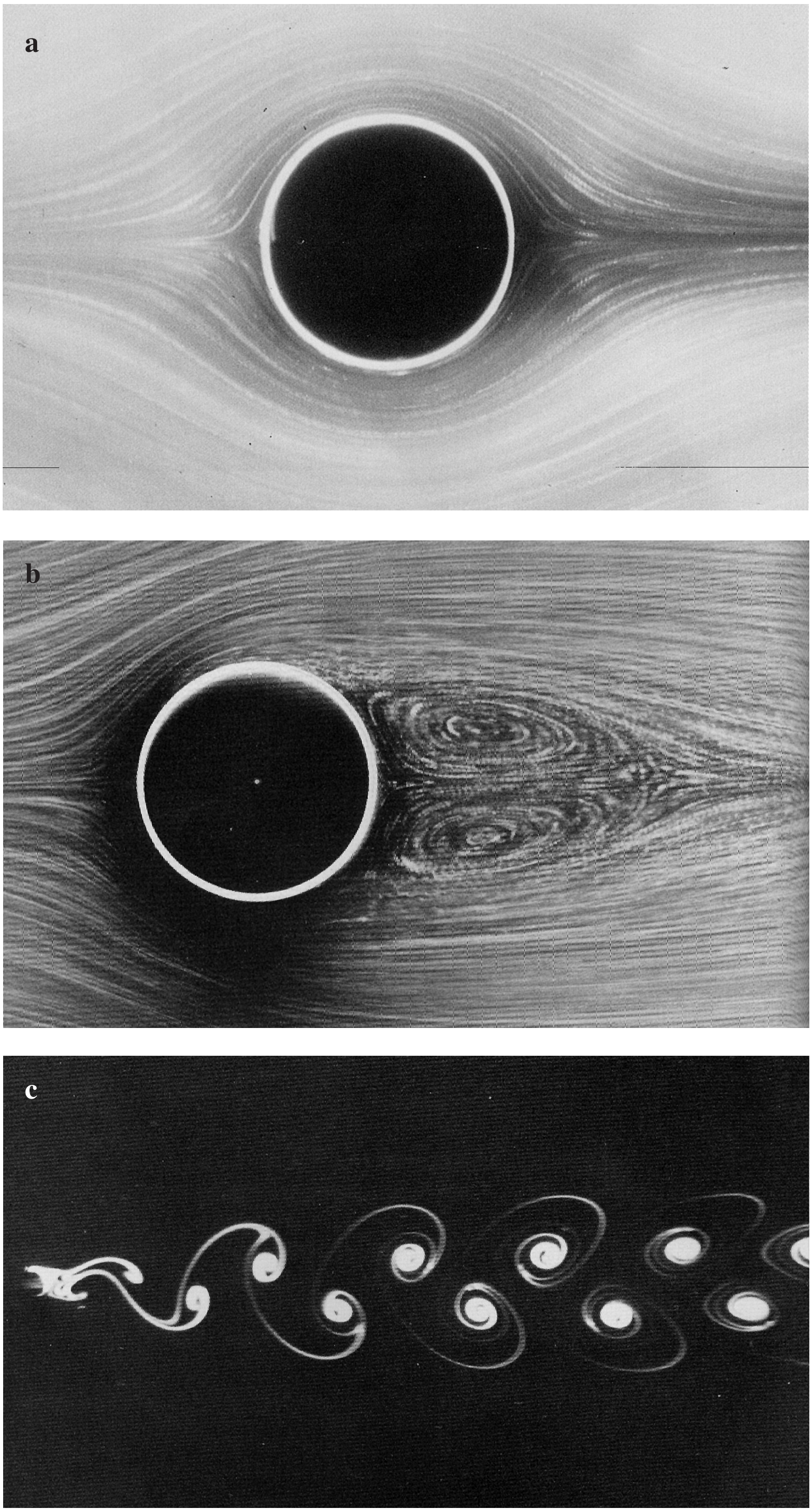

FIG. 8. – Photographs of streamlines (a, b) or streaklines (c) for steady flow past a circular cylinder at different values of the Reynolds number (M.Van Dyke, 1982): (**a**) $Re \ll 1$, (**b**) $Re \approx 26$, (**c**) $Re \approx 105$.

segment of the flagellum moving perpendicular to itself relative to the water as is generated by the same segment moving parallel to itself. This fact forms the basis of *resistive force theory* for flagellar propulsion, which is a simple and reasonably accurate model for the analysis of flagellar locomotion.

Vorticity

The dynamics of fluid flow can often be most deeply understood in terms of the *vorticity*, defined by equation (11) and representing the local rotation of fluid elements. High velocity gradients correspond to high vorticity (see fig. 4). If we take the curl of every term in the Navier-Stokes equation we obtain the following vorticity equation (in vector notation):

$$\frac{\partial \omega}{\partial t} + (\mathbf{u}.\nabla)\omega = (\omega.\nabla)\mathbf{u} + \nu\nabla^2\omega + \frac{1}{\rho^2}\nabla\rho \wedge \nabla p \qquad (35)$$

where $\nu = \mu/\rho$ is the *kinematic viscosity* of the fluid (assumed constant). This equation tells us that the vorticity, evaluated at a fluid element locally parallel to $\boldsymbol{\omega}$, changes, as that element moves, as a result of three effects, each represented by one of the terms on the right hand side of (35). The first term can be shown to be associated with rotation and stretching (or compression) of the fluid element, so that the direction of $\boldsymbol{\omega}$ remains parallel to the original fluid element, and increases in proportion as the length of that element changes. Such *vortex-line stretching* is a dominant effect in the generation and maintenance of turbulence. It is totally absent in a two-dimensional (2D) flow in which there is no velocity component in one of the coordinate directions (say z) and the variables are independent of z. The second term represents the effect of viscosity, and is *diffusion*-like in that vorticity tends to spread out from elements where it is high to those where it is low. The last term comes about only in non-uniform (e.g. stratified) fluids, and can be important in some oceanographic situations.

It can also be shown that, in a flow started from rest, no vorticity develops anywhere until viscous diffusion has had an effect there. As we shall see, the only *source* of vorticity, in such a flow and in the absence of the last term in (35), occurs at solid boundaries on account of the no-slip condition.

Higher Reynolds number.

It is convenient now to restrict attention to a 2D flow of a homogeneous fluid past a 2D body such as a circular cylinder (fig. 8). In such a 2-D flow, with velocity components $\mathbf{u} = (u,v,0)$, functions of x,y and t, the vorticity is entirely in the third, z, direction, and is given by

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}.$$

There is no vortex-line stretching, and the only effect which can generate vorticity anywhere is viscosity. Let us suppose that the uniform stream at infinity is switched on from rest at the initial instant. Initially there is no vorticity anywhere, and the initial *irrotational* velocity field is easy to calculate. It satisfies all the governing equations and all boundary conditions *except* the no-slip condition at the cylinder surface. The predicted slip velocity therefore generates an infinite velocity gradient $\partial u/\partial y$ and hence a thin sheet, of infinite vorticity at the cylinder surface. Because of viscosity, this immediately starts to diffuse out from the surface. At low values of Re, when viscosity is dominant and the convective term $(\mathbf{u}.\nabla)\boldsymbol{\omega}$ in (35) is negligible, the diffusion is rapid, and vorticity spreads out a long way in all directions. An eventual steady state is set up in which the flow is almost totally symmetric front-to-back (fig. 8a); unlike the spherical case, the drag coefficient is not quite inversely proportional to the Reynolds number:

$$C_D = \frac{8\pi}{Re\log(7.4/Re)}.$$

At somewhat higher values of Re, the $(\mathbf{u}.\nabla)\boldsymbol{\omega}$ term is not totally negligible, and once vorticity has reached any particular fluid element it tends to be carried along by it as well as diffusing on to other elements. Hence a front-to-back asymmetry develops. For Re greater than about 5 the flow actually separates from the wall of the cylinder, forming two slowly recirculating flow regions (eddies) at the rear. At still higher Re, it is observed that the eddies tend to break away alternately from the two sides of the cylinder, usually at a well-defined frequency equal to about 0.42 U_∞/a for $Re \geq 600$, and steady flow is no longer possible. At higher Re the wake becomes *turbulent* (i.e. random and three-dimensional) and at $Re \approx 2 \times 10^5$ the flow on the cylinder surface becomes turbulent.

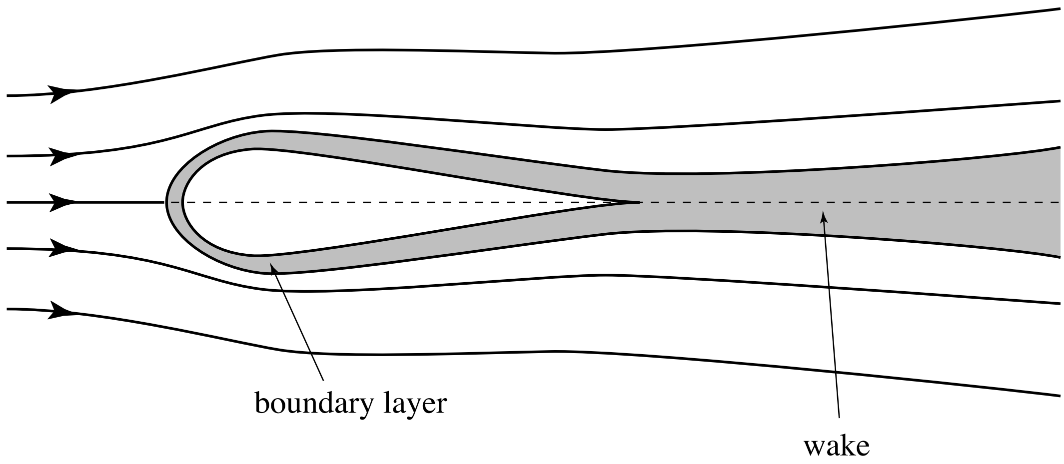

FIG. 9. – Sketch of boundary layer and wake for steady flow at high Reynolds number past a symmetric streamlined body.

Steady flows at relatively high Reynolds number do seem to be possible past streamlined bodies such as a wing (or a fish dragged through the fluid), see fig. 9. Diffusion causes vorticity to occupy a (boundary) layer of thickness $(\nu t)^{1/2}$ after time t. However, even a fluid element near the leading edge at first will have been swept off downstream past the trailing edge after a time $t = L/U_\infty$, where L is the length of the wing chord. Hence the greatest thickness that the boundary layer on the body can have is

$$\delta_s = \left(\nu L / U_\infty\right)^{1/2}, \tag{36}$$

and it is easy to see that a steady state can develop everywhere on the body, with a boundary layer of thickness up to δ_s, and a thin wake region, also containing vorticity, downstream. Note that the boundary layer of vorticity remains thin compared with the chord length if $\delta_s << L$, i.e. $Re >>1$. In that case (and only then) neglecting viscosity altogether, and forgetting about the boundary layer, is accurate enough, except in calculating the drag.

Drag on a symmetric body at large Reynolds number. In order to estimate the force on a body it is necessary to work out the distribution of pressure

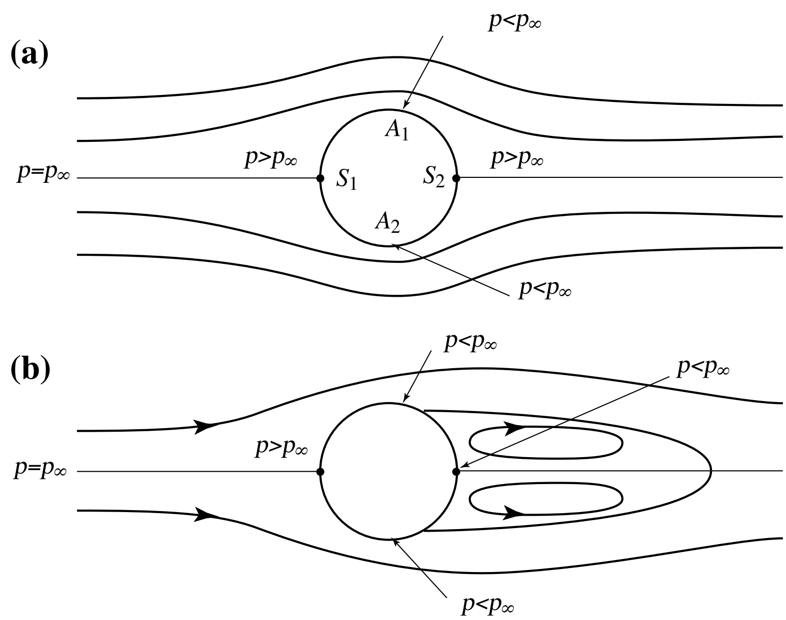

FIG. 10. – Sketch of streamlines and pressures for flow past a circular cylinder. (**a**) Idealised flow of a fluid with no viscosity; (**b**) separated flow at fairly high Reynolds number in a viscous fluid.

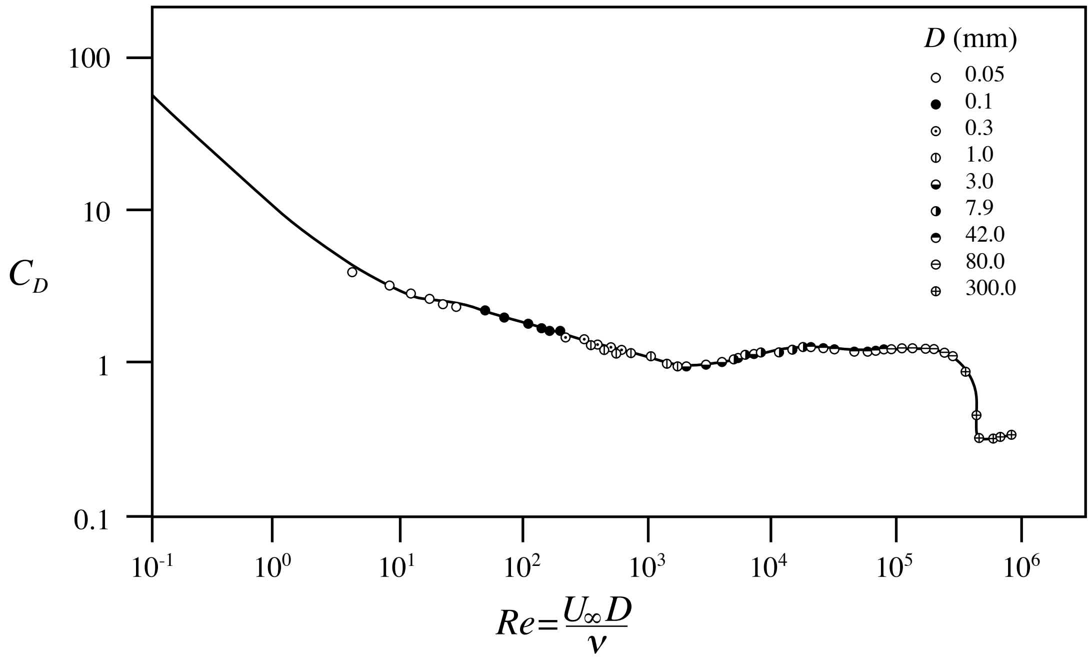

FIG. 11. – Log-log plot of drag coefficient versus Reynolds number for steady flow past a circular cylinder. [The sharp reduction in C_D at $Re \approx 2 \times 10^5$ is associated with the transition to turbulence in the boundary layer]. Redrawn from Schlichting (1968).

round the body. In a steady flow of constant density fluid in which viscosity is unimportant (e.g. outside the boundary layer and wake of a body) equation (16d) can be integrated to give the result that the quantity

$$p + pgz + \frac{1}{2}\rho|\mathbf{u}|^2 = \text{constant} \tag{37a}$$

along streamlines of the flow. Here z is measured vertically upwards and $|\mathbf{u}|$ is the total fluid speed. This result is equivalent to the Newtonian principle of conservation of energy; equation (37a) is called *Bernoulli's equation*. If we forget about the gravitational contribution, replacing $p + pgz$ by the effective pressure p_e (eq. 21), equation (37a) becomes

$$p_e = \text{constant} - \frac{1}{2}\rho|\mathbf{u}|^2; \tag{37b}$$

henceforth we just write p for p_e. If the fluid speeds up, the pressure falls, and vice versa, which is intuitively obvious since a favourable pressure gradient is clearly required to give fluid elements positive acceleration.

In the case of flow past a symmetric body, (fig. 10a), all streamlines start from a region of uniform pressure (p_∞ say) and uniform velocity (U_∞), so the constant in (37b) is the same for all streamlines, p_∞+1/2U^2_∞. If viscosity were really negligible, then the flow round a circular cylinder would be symmetric (fig. 10a). At the front *stagnation point* S_1, the point of zero velocity where the streamline dividing flow above from flow below impinges, the pressure is high ($p = p_\infty$+1/2ρU^2_∞), and this high pressure is balanced by an equally high pressure at the rear stagnation point S_2. The pressure at the sides (A_1, A_2) is low ($p = p_\infty$–3/2U^2_∞). The net effect is that the hydrodynamic force on the cylinder is zero.

In a viscous fluid, as stated above, there is a thin boundary layer on the front half, in which the velocity falls from a large value to zero, so the pressure distribution is similar to that described above; however the flow separates on the rear half and things are very different. The reason for the separation is that the adverse pressure gradient (the pressure rise), from A_1 to S_2 say, causes the low velocity in the boundary layer to tend to reverse its direction, and it is observed that separation occurs as soon as flow reversal takes place. In the separated flow region (fig. 10b) the fluid velocity is low and the pressure remains close to its value at the sides. Thus there is a front-to-back pressure difference proportional to ρU^2_∞, and the drag coefficient C_D (eq. 28) is approximately constant, independent of Re as long as Re is large (see fig. 11). The direct contribution of tangential viscous stresses to the drag is negligibly small, although it is the presence of viscosity which causes the flow separation in the first place.

Lift. For a symmetric streamlined body (fig. 9) flow separation occurs only very near the trailing edge, and direct viscous drag is more important. However, if such a streamlined body (or wing) is tilted so that the oncoming flow makes an angle of incidence with its centre plane, viscosity again has an important effect. In general, a non-viscous flow past a wing at incidence would turn sharply round the trailing edge, where the velocity would be extremely high and the pressure extremely low (fig. 12a). As the flow starts up from rest, viscosity causes separation at the corner, a concentrated vortex is shed and left behind, and thereafter the flow is forced to come tangentially off the trailing edge: the *Kutta-condition* (fig. 12b). In order to achieve this tangential flow, the velocity on top of the wing must increase and the velocity below must decrease. It follows from Bernoulli's equation that the pressure above the wing must fall, and that below rise, so a transverse force is generated. This is call *lift* and keeps aircraft and birds in the air against gravity. The magnitude of the lift is also represented by a *lift coefficient* C_L :

$$L = \frac{1}{2}U_\infty^2 SC_L, \qquad (38)$$

where S is the horizontal area of the wing. Like C_D, C_L is approximately independent of Re for large Re .

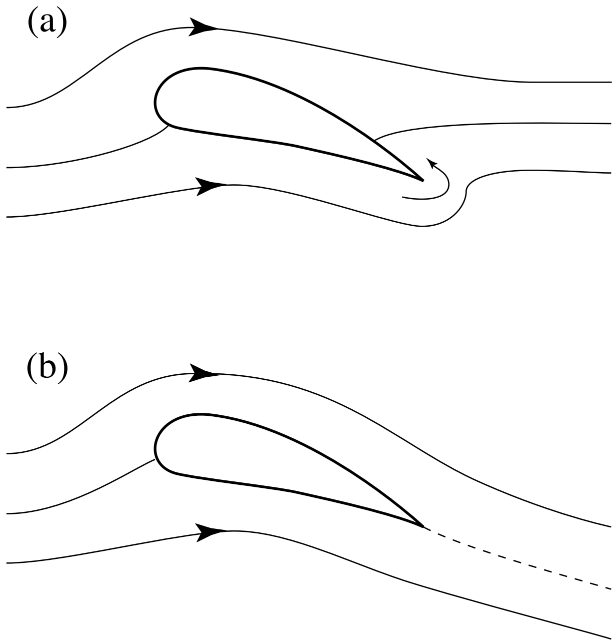

FIG. 12. – Flow past a streamlined body at incidence. (**a**) Idealised flow of a fluid with no viscosity - large velocity and pressure gradient round the trailing edge. (**b**) In a viscous fluid the flow must come smoothly off the trailing edge, which explains the generation of lift (see text).

Added mass. We have seen that the force on a body in an inviscid fluid is zero when the flow is steady. When the flow is unsteady, however, the force is non-zero, because accelerating the body relative to the fluid requires that the fluid also has to be accelerated. Thus the body exerts a force on the fluid and so, by Newton's third law, the fluid exerts an equal and opposite force on the body. In all cases, this force is equal in magnitude to the acceleration of the body relative to the fluid multiplied by the mass of fluid displaced by the body (ρV in the notation of eq. 20) multiplied by a constant, say β:

$$F=\beta\rho V\, dU/dt. \qquad (39)$$

For a sphere, $\beta = 0.5$; for a circular cylinder, $\beta = 1$. The quantity $\beta\rho V$ is call the *added mass* of the body in question (recall that ρ is the *fluid* density). The corresponding force, given by (39), is called the *acceleration reaction*, or the *reactive force*.

Fish swimming. We have seen that flagellates such as spermatozoa swim by sending bending waves down their tails, and thrust is generated through the viscous, resistive force. Inertia is negligible because the Reynolds number is small. For most fish, the Reynolds number is large, but nevertheless many fish also swim by sending a bending wave down their bodies and tails. In this case, however, thrust is generated primarily by the reactive force associated with the sideways acceleration of the elements of fluid as they pass down the animal (relative to a frame of reference fixed in the fish's nose). Lighthill has developed a simple, reactive-force model for fish swimming.

Flow in the open ocean

Water waves

The most obvious dynamical feature of the ocean, to even a casual observer, is the presence of surface *waves*, of a variety of lengths and heights. Waves are mainly generated as a result of stresses exerted by the wind, although they can also arise through the impact or relative motion of foreign bodies such as rain drops or ships. Once generated, however, waves can propagate over large distances and persist for long times, unaffected by the atmosphere or solid bodies.

In a periodic wave motion, all fluid elements affected by it experience oscillations. Like all oscillations, such as that of a simple pendulum, these oscillations come about as an interaction between a *restoring force*, tending to restore a particle to a nearby equilibrium position, and *inertia*, which causes the particle to overshoot each time it reaches its equilibrium position (in real systems there is also some viscous *damping*, which causes the amplitude of the oscillations to die out after a long time, if there is no further stimulation; we ignore damping here). In the case of a simple pendulum (a mass suspended by a light string) the equilibrium state is one in which the string is vertical and the mass at rest, the restoring force is gravity and the inertia is the momentum of the mass itself. In the case of water waves, the equilibrium state has the free surface horizontal, the restoring force is again gravity (except for small wavelengths, when surface tension is also important) and the inertia is the momentum of the fluid. Viscosity is negligible because there are no solid boundaries generating vorticity.

In an oscillation of small amplitude, every particle exhibits *simple harmonic motion*: its vertical displacement, say Y, from equilibrium, varies with time according to the differential equation

$$\frac{d^2Y}{dt^2} + \omega^2 Y = 0. \qquad (40)$$

The general solution for Y is a sinusoidal oscillation of the form

$$Y = A\cos\left(\omega t - \phi\right)$$

where A and ϕ are constants (determined by initial conditions), the amplitude and phase respectively, and ω is the *angular frequency* of the oscillation (the frequency in Hertz is $\omega/2\pi$). In the case of a simple pendulum, $\omega = (g/l)^{1/2}$ where l is the length of the string. In the case of simple water waves of wave length $\lambda = 2\pi/k$ (k is the *wave number*), in an ocean whose depth is much greater than λ, we have

$$\omega = (gk)^{1/2} \qquad (41)$$

as long as surface tension is negligible.

Suppose a parallel-crested (one-dimensional) train of such waves is propagating in the x-direction. Then the displacement of the free surface will be given by

$$\eta = A\cos(\omega t - kx - \phi), \qquad (42)$$

again for constant A and ϕ. The speed of propagation of the wave crests, or *phase velocity*, is

$$c = \frac{\omega}{k} = \left(\frac{g}{k}\right)^{1/2}. \qquad (43)$$

Thus long waves (small k) travel more rapidly than short waves (large k). This explains why, when the waves are generated by a localised disturbance, such as a storm at sea, or a stone dropped in a pond, the longer waves (swell) arrive at the shore first. In this case, the wave front travels at a different speed, called the group velocity, c_g:

$$c_g = \frac{d\omega}{dk} = \frac{1}{2}\left(\frac{g}{k}\right)^{1/2}, \qquad (44)$$

so that wave crests, travelling faster, appear to arise at the back of the packet of waves, and to disappear at the front.

When a water wave propagates, with its free surface given by (42), fluid elements at and below the surface move in circular paths, and the amplitude of their motion falls off exponentially with depth below the surface: the amplitude is proportional to Ae^{kz} when the undisturbed surface is at $z = 0$. Thus the amplitude is negligibly small at a depth of only half a wavelength ($kz = -\pi$). This explains why the theory of waves in very deep water works well in relatively shallow water, too, with depth h greater than half a wavelength. When the waves are very long, or the water very shallow, equation (41) is replaced by

$$\omega = \left(gk \tanh kh\right)^{1/2} \qquad (45)$$

Small amplitude wave theory is very useful, because the equations are linear and a general motion can be made up from the addition of many sinusoidal components such as (42) (a Fourier series or transform). At larger amplitudes, nonlinear effects become important and the theory becomes less general, although many interesting and important phenomena arise, such as wave breaking.

Internal waves

Although the water in the ocean is effectively incompressible, it does not have uniform density because it is *stratified* on account of the variation with depth of the pressure and, to a lesser extent, the temperature and the salinity. The temperature/density distribution is marked usually by one or more thermoclines, in which the density gradient is steeper than elsewhere. Whether the density gradient is uniform or locally sharp, less dense fluid sits, in equilibrium, above denser fluid. A disturbance to this state causes some heavy fluid elements to rise above their original level, and some light ones to fall below. As in the case of surface waves, gravity then provides a restoring force and *internal gravity waves* can propagate. As for surface waves, a relation can be calculated between the frequency and the wave number of such waves. For example, if there is a sharp interface between two deep regions of fluid with densities ρ_1 (above) and ρ_2, then equation (41) is replaced by

$$\omega^2 = gk\left[(\rho_2 - \rho_1)/(\rho_2 + \rho_1)\right]. \qquad (46)$$

This can be seen to give much lower frequencies than (41) if $(\rho_2 - \rho_1)$ is not large: if $\rho_2 - \rho_1 = 0.1\ \rho_2$, then the frequency given by (46) is 4.4 times smaller than that given by (41) (with $\rho_2 = \rho$). The propagation speed is correspondingly smaller, too.

When the density gradient is uniform, with

$$\frac{g}{\rho}\frac{d\rho}{dz} = -N^2, \qquad (47)$$

where N is a constant with the dimensions of a frequency (the *Brunt -Väisälä* frequency), the situation is a bit more complicated, because internal waves do not have to propagate horizontally. Indeed, a wave whose crests propagate at an angle θ to the horizontal, so that the displacement of a fluid element is given by

$$y = A\cos\left[\left(\omega t - k(x\cos\theta + z\sin\theta)\right)\right],$$

has a frequency ω given by

$$\omega = N\cos\theta\,. \qquad (48)$$

However, the group velocity (velocity of a wave front, or of energy propagation) is perpendicular to the phase velocity, and in this case is given by the vector

$$\mathbf{c}_g = \frac{N}{k}\sin\theta(\sin\theta, 0, -\cos\theta). \qquad (49)$$

Rotating fluids: geostrophic flows

Gravity waves are (mostly) small-scale phenomena for which the rotation of the earth is unimportant. That is not the case with ocean currents and the large-scale circulation of the oceans. To analyse such motions, it is necessary to recognise that the natural frame of reference is fixed in the rotating earth, and the governing equations of motion have to be changed accordingly. If viscosity is neglected, the equation of motion of a fluid in a frame of reference rotating with constant angular velocity Ω becomes (in place of (16d)):

$$\rho\frac{D\boldsymbol{u}}{Dt} + \rho\Omega \wedge \boldsymbol{u} = \rho\mathbf{g} - \nabla p. \qquad (50)$$

Here **g** has been modified to include the small "centrifugal force" term, and we could also incorporate it into the pressure using (21). The additional term $\rho\Omega \wedge \boldsymbol{u}$ is called the *Coriolis force*.

Time does not permit a thorough investigation of the dynamics of rotating fluids. We consider only a flow in which the Coriolis force is much larger than the other inertia terms and therefore must by itself balance the gradient in (effective) pressure: a *geostrophic flow*. For such a flow, (50) reduces to

$$\rho\Omega \wedge \boldsymbol{u} = -\nabla p. \qquad (51)$$

Suppose the flow is horizontal: $\mathbf{u} = (u, v, 0)$, with z vertically upwards again. Then the horizontal components of the pressure gradient are given by

$$\frac{\partial p}{\partial x} = -\Omega_v v, \frac{\partial p}{\partial y} = +\Omega_v u \qquad (52)$$

where Ω_v is the vertical component of the earth's angular velocity (total angular velocity multiplied by the sine of the latitude). The pressure gradient is perpendicular to the velocity, or vice versa, indicating that if there is a horizontal pressure gradient, the corresponding geostrophic flow will be perpendicular to it. This explains why the wind goes anticlockwise round atmospheric depressions in the northern hemisphere (clockwise in the southern hemisphere). Similar flows occur in the oceans, although the barriers formed by the continents are impermeable, unlike in the atmosphere.

The condition for a steady flow to be geostrophic is that the inertia term $(\boldsymbol{u}.\nabla)\boldsymbol{u}$ should be small compared with the Coriolis term. Thus if U is a typical velocity magnitude, and L a length scale for the flow, the geostrophic approximation will be a good one if

$$\frac{U^2}{L} << \Omega_v U,$$

i.e. the *Rossby number* should be small:

$$\frac{U^2}{\Omega_v L} << 1. \qquad (53)$$

If the Rossby number is large, the earth's rotation can be neglected. Note that the Rossby number is always large at the equator, where $\Omega_v = 0$.

Hydrodynamic instability

A smooth, laminar flow becomes turbulent as a result of hydrodynamic instability. Small, random perturbations are inevitably present in any real system; if they die away again, the flow is stable, but if they grow large, the original flow becomes unrecognisable and is unstable. Usually, steady flows which are slow or weak enough are stable, but they become unstable above some critical speed or strength.

The way to investigate instabilities mathematically is to assume that the disturbances to the steady state are very small and to linearise the equations accordingly. Thus, if the steady state velocity, pressure and density are given by $\mathbf{u}_0(\mathbf{x})$, $p_0(\mathbf{x})$ and $\rho_0(\mathbf{x})$ (all functions of position, in general) it is postulated that, with the perturbation, we have

$$\mathbf{u} = \mathbf{u}_0(\mathbf{x}) + \mathbf{u}'(\mathbf{x},\ t),$$
$$p = p_0(\mathbf{x}) + p'(\mathbf{x},\ t),$$
$$\rho = \rho_0(\mathbf{x}) + \rho'(\mathbf{x},\ t)$$

where $\mathbf{u}'$, p' and ρ' are small. Then these are substituted into the governing equations, and terms involving squares or products of small quantities are neglected, so the equations are linearised. For example, equation (7) which, with (3b), is

$$\frac{\partial \rho}{\partial t} + \mathbf{u}.\nabla\rho = 0,$$

becomes

$$\frac{\partial \rho'}{\partial t} + \mathbf{u}_0.\nabla\rho' + \mathbf{u}'.\nabla\rho_0 = 0 \qquad (54)$$

and the nonlinear term $\mathbf{u}'\nabla\rho'$ is neglected. After linearisation, it is usually possible to think of the disturbance as made up of many *modes* in which the variables depend sinusoidally on one or more space coordinates and exponentially on time, e.g.

$$\rho' = f(z)\exp\{i(kx + ly) + \sigma t\} \qquad (55)$$

(cf 42), where $i = \sqrt{-1}$ and we are using complex number notation. Such terms are substituted into the equations, and it turns out that a solution of the supposed form exists only if σ takes a particular value. If that value has negative (or zero) real part, the disturbance dies away (or oscillates at constant amplitude); if it has positive real part it grows exponentially, indicating instability. If *any* disturbance of the form (55) (i.e. for any values of k and l) grows, then the flow is unstable, because in general all disturbances are present, infinitesimally, at first.

Consider, for example, the case of two fluids of different densities, one on top of the other. We have seen that the frequency of a disturbance of wavenumber k is given by equation (46) if ρ_2(the density of the lower fluid) is greater than ρ_1. However, if $\rho_1 > \rho_2$, ω^2 as given by (46) is negative. But if we replace ω by $i\sigma$, σ^2 is positive, σ is real, and the oscillation $\cos\omega t$ can be written as $1/2(e^{\sigma t} + e^{-\sigma t})$.Thus exponential growth is predicted. Hence the interface between a dense fluid and a less dense fluid below it is unstable.

A similar analysis can be performed for a continuous density distribution, denser on top, caused by a temperature gradient, say, in a fluid heated from below. In this case the diffusion of heat (and hence density) must be allowed for, as well as conservation of fluid mass and momentum. For example, a horizontal layer of fluid, contained between two rigid horizontal planes, distance h apart and maintained at temperatures T_0 (top) and $T_0 + \Delta T$ (bottom) is unstable if the temperature difference ΔT is large enough. More precisely, instability occurs if a dimensionless parameter called the *Rayleigh number Ra* exceeds the critical value of 1708, where

$$Ra = \frac{g\alpha\Delta T h^3}{\nu\kappa} \qquad (56)$$

Here α, ν and κ are fluid properties, the coefficient of expansion, the kinematic viscosity (μ/ρ) and the thermal diffusivity respectively. When instability occurs, for values of Ra not much greater than 1708, the resulting motion is a regular array of usually hexagonal cells (fig. 13), with fluid flow up in the centre of the cells and down at the edges. Such a motion is an example of *thermal convection*, called *Rayleigh-Benard* convection. When Ra is much higher than 1708, the cells themselves become unstable, the convection becomes very complicated and eventually turbulent.

Rayleigh-Benard convection is an example in which instability of the original steady state leads to another, regular, steady motion which itself goes

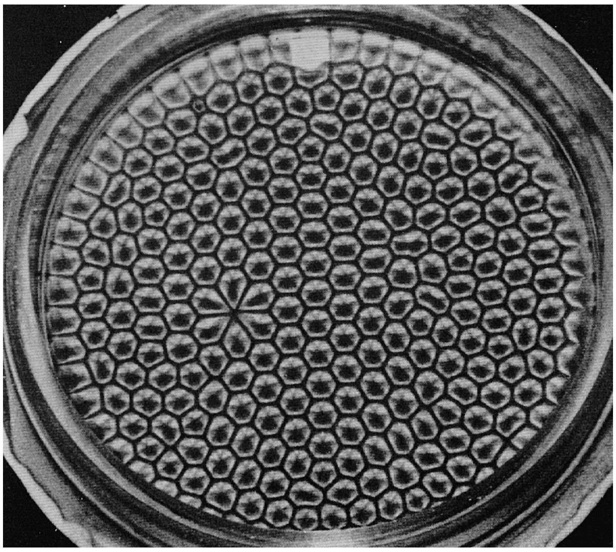

FIG. 13. – Photograph of convection pattern for Rayleigh-Benard convection in a layer of fluid heated from below. (Koschmieder, 1974).

unstable as Ra is increased, and turbulence results only after a whole sequence of such instabilities, or bifurcations. Other systems do not seem to have intermediate stable steady states, but there is a rapid transition from laminar to turbulent flow when critical conditions are passed. Perhaps the most familiar and important of such flows are uni-directional (or approximately so) *shear flows*, such as that depicted in fig. 4. Examples are flow in a straight pipe and flow in the boundary layer on a rigid body or in the shear layer at the edge of the recirculation behind it. Flow in a circular pipe of diameter D normally becomes turbulent when the Reynolds number

$$Re = \frac{D\bar{u}}{\nu},$$

where $\bar{u}$ is the cross-sectionally averaged velocity, exceeds a critical value of just over 2000. Flow in a boundary layer on a thin flat plate (an approximation to a streamlined body) becomes unstable when the Reynolds number based on the free stream velocity and the boundary layer thickness δ_s (eq. 36) exceeds about 244. Flow in a shear layer is more unstable still, associated with the fact that the velocity profile contains an inflection point.

When numbers are put in to formulae such as those quoted above, it becomes clear that oceanic flows are necessarily turbulent. Hence the existence of this course.

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